Relations among Fourier Representations
Mid-term Examination #3

Wednesday, April 28, 7:30-9:30pm.

No recitations on the day of the exam.

Coverage: Lectures 1–20
          Recitations 1–20
          Homeworks 1–11

Homework 11 will not collected or graded. Solutions will be posted.

Closed book: 3 pages of notes (8½ × 11 inches; front and back).

Designed as 1-hour exam; two hours to complete.

Review sessions during open office hours.
Fourier Representations

We’ve seen a variety of Fourier representations:

- CT Fourier series
- CT Fourier transform
- DT Fourier series
- DT Fourier transform

Today: relations among the four Fourier representations.
Four Fourier Representations

We have discussed four closely related Fourier representations.

### DT Fourier Series

\[
\begin{align*}
    a_k &= a_{k+N} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j \frac{2\pi}{N} kn} \\
    x[n] &= x[n + N] = \sum_{k=\langle N \rangle} a_k e^{j \frac{2\pi}{N} kn}
\end{align*}
\]

### DT Fourier transform

\[
\begin{align*}
    X(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \\
    x[n] &= \frac{1}{2\pi} \int_{\langle 2\pi \rangle} X(e^{j\Omega}) e^{j\Omega n} d\Omega
\end{align*}
\]

### CT Fourier Series

\[
\begin{align*}
    a_k &= \frac{1}{T} \int_T x(t) e^{-j \frac{2\pi}{T} kt} dt \\
    x(t) &= x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j \frac{2\pi}{T} kt}
\end{align*}
\]

### CT Fourier transform

\[
\begin{align*}
    X(j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \\
    x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega
\end{align*}
\]
Four Types of “Time”

discrete vs. continuous (↑↓) and periodic vs aperiodic (↔)

DT Fourier Series

\[
\begin{array}{c}
\ldots \\
\downarrow \\
\ldots
\end{array}
\quad n
\]

DT Fourier transform

\[
\begin{array}{c}
\ldots \\
\uparrow \\
\ldots
\end{array}
\quad n
\]

CT Fourier Series

\[
\begin{array}{c}
\ldots \\
\downarrow \\
\ldots
\end{array}
\quad t
\]

CT Fourier transform

\[
\begin{array}{c}
\ldots \\
\uparrow \\
\ldots
\end{array}
\quad t
\]
Four Types of “Frequency”

discrete vs. continuous (↔) and periodic vs aperiodic (↑)

DT Fourier Series

\[ \frac{2\pi}{N} k \]

DT Fourier transform

\[ \Omega \]

CT Fourier Series

\[ \frac{2\pi}{T} k \]

CT Fourier transform

\[ \omega \]
Relations among Fourier Representations

Different Fourier representations are related because they apply to signals that are related.

DTFS (discrete-time Fourier series): periodic DT
DTFT (discrete-time Fourier transform): aperiodic DT
CTFS (continuous-time Fourier series): periodic CT
CTFT (continuous-time Fourier transform): aperiodic CT

\[ N \to \infty \]
\[ T \to \infty \]

interpolate
sample

periodic extension

periodic DT
DTFS

aperiodic DT
DTFT

interpolate
sample

periodic extension

periodic CT
CTFS

aperiodic CT
CTFT

interpolate
sample

periodic extension
A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

Series: represent periodic signal as weighted sum of harmonics

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}; \quad \omega_0 = \frac{2\pi}{T} \]

The Fourier transform of a sum is the sum of the Fourier transforms:

\[ X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) \]

Therefore periodic signals can be equivalently represented as Fourier transforms (with impulses!).
A periodic signal can be represented by a Fourier series or by an equivalent Fourier transform.

\[ x(t) = x(t + T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \leftrightarrow \]

**Fourier Series**

![Fourier Series Diagram]

**Fourier Transform**

![Fourier Transform Diagram]
Explore other relations among Fourier representations.

Start with an aperiodic CT signal. Determine its Fourier transform.

Convert the signal so that it can be represented by alternate Fourier representations and compare.
Start with the CT Fourier Transform

Determine the Fourier transform of the following signal.

\[ x(t) \]

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t} \, dt \]

Could calculate Fourier transform from the definition.
Start with the CT Fourier Transform

Determine the Fourier transform of the following signal.

\[ x(t) \]

\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{j\omega t} dt \]

Could calculate Fourier transform from the definition.

Easier to calculate \( x(t) \) by convolution of two square pulses:
Start with the CT Fourier Transform

The transform of \( y(t) \) is \( \frac{2\sin(\omega/2)}{\omega} \).

\[
y(t) \quad Y(j\omega)
\]

\[
-\frac{1}{2} \quad \frac{1}{2} 
\]

so the transform of \( x(t) = (y * y)(t) \) is \( X(j\omega) = Y(j\omega) \times Y(j\omega) \).
Relation between Fourier Transform and Series

What is the effect of making a signal periodic in time?

Find Fourier transform of periodic extension of $x(t)$ to period $T = 4$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$

Could calculate $Z(j\omega)$ for the definition ... ugly.
Relation between Fourier Transform and Series

Easier to calculate $z(t)$ by convolving $x(t)$ with an impulse train.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$

where

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t + 4k)$$

Then

$$Z(j\omega) = X(j\omega) \times P(j\omega)$$

We already know $P(j\omega)$: it’s also an impulse train!
Convolving in time corresponds to multiplying in frequency.

\[ X(j\omega) \]

\[ P(j\omega) \]

\[ Z(j\omega) \]
Relation between Fourier Transform and Series

The Fourier transform of a periodically extended function is a discrete function of frequency $\omega$.

$$z(t) = \sum_{k=-\infty}^{\infty} x(t + 4k)$$
The weight (area) of each impulse in the Fourier transform of a periodically extended function is \(2\pi\) times the corresponding Fourier series coefficient.

\[
Z(j\omega) = \frac{1}{2\pi} \frac{\pi}{2} \leq \omega \leq \frac{\pi}{2} \quad a_k = \frac{1}{4} \leq k \leq 1
\]
The effect of periodic extension of $x(t)$ to $z(t)$ is to sample the frequency representation.

\[ X(j\omega) \]

\[ Z(j\omega) \]

\[ a_k \]

\[ k = -1, 0, 1 \]
Relation between Fourier Transform and Series

Periodic extension of a CT signal produces a discrete function of frequency.

Periodic extension

\[ = \text{convolving with impulse train in time} \]
\[ = \text{multiplying by impulse train in frequency} \]
\[ \rightarrow \text{sampling in frequency} \]

[Diagram showing the relationship between periodic and aperiodic DT and CT signals]

\[ N \rightarrow \infty \] for periodic DT to aperiodic DT

\[ T \rightarrow \infty \] for periodic CT to aperiodic CT

(periodic extension)

(interpolate) \rightarrow \text{(sampling in frequency)}
Four Types of “Time”

discrete vs. continuous (↑) and periodic vs aperiodic (↔)

**DT Fourier Series**

**DT Fourier transform**

**CT Fourier Series**

**CT Fourier transform**
Four Types of “Frequency”

discrete vs. continuous (↔) and periodic vs aperiodic (↕)

**DT Fourier Series**

\[ \frac{2\pi}{N} k \]

**DT Fourier transform**

\[ \Omega \]

**CT Fourier Series**

\[ \frac{2\pi}{T} k \]

**CT Fourier transform**

\[ \omega \]
Relations among Fourier Representations

Compare to sampling in time.

\[ N \to \infty \]

interpolate

periodic extension

\[ T \to \infty \]

periodic extension
Relations between CT and DT transforms

Sampling a CT signal generates a DT signal.

\[ x[n] = x(nT) \]

Take \( T = \frac{1}{2} \).

What is the effect on the frequency representation?
Relations between CT and DT transforms

We can generate a signal with the same shape by multiplying \( x(t) \) by an impulse train with \( T = \frac{1}{2} \).

\[
x_p(t) = x(t) \times p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t + kT)
\]
Relations between CT and DT transforms

Multiplying $x(t)$ by an impulse train in time is equivalent to convolving $X(j\omega)$ by an impulse train in frequency (then $\div 2\pi$).
The Fourier transform of the “sampled” signal $x_p(t)$ is periodic in $\omega$ with period $4\pi$. 

\[ x_p(t) \]

\[ X_p(j\omega) \]

-4\pi \quad 4\pi
The Fourier transform of the “sampled” signal $x_p(t)$ has the same shape as the DT Fourier transform of $x[n]$. 

$$x[n]$$

$$n$$

$$-1 \quad 1$$

$$X(e^{j\Omega})$$

$$\Omega$$

$$-2\pi \quad 2\pi$$
DT Fourier transform

The CT Fourier transform of a “sampled” signal \((x_p(t))\) is equal to the DT Fourier transform of the samples \((x[n])\) where \(\Omega = \omega T\), i.e.,

\[ X(j\omega) = X(e^{j\Omega}) \bigg|_{\Omega = \omega T}. \]

\[ \Omega = \omega T = \frac{1}{2} \omega \]
Relation between CT and DT Fourier transforms

The CT Fourier transform of a “sampled” signal \( x_p(t) \) is equal to the DT Fourier transform of the samples \( x[n] \) where \( \Omega = \omega T \), i.e.,

\[
X(j\omega) = X(e^{j\Omega}) \bigg|_{\Omega = \omega T}.
\]
Four Types of “Time”

discrete vs. continuous (↑↓) and periodic vs aperiodic (↔)

**DT Fourier Series**

---

**CT Fourier Series**

---

**DT Fourier transform**

---

**CT Fourier transform**

---
Four Types of “Frequency”

discrete vs. continuous (↔) and periodic vs aperiodic (↕)

DT Fourier Series

CT Fourier Series

DT Fourier transform

CT Fourier transform
Periodic extension of a DT signal is equivalent to convolution of the signal with an impulse train.

\[ x[n] \]

\[ p[n] \]

\[ xp[n] = (x * p)[n] \]
Relation Between DT Fourier Transform and Series

Convolution by an impulse train in time is equivalent to multiplication by an impulse train in frequency.

\[ X(e^{j\Omega}) \]

\[ P(e^{j\Omega}) \]

\(\cdots\) \(\cdots\)

\[ X_p(e^{j\Omega}) \]

\(\cdots\) \(\cdots\)
Relation Between DT Fourier Transform and Series

Periodic extension of a discrete signal \((x[n])\) results in a signal \((x_p[n])\) that is both periodic and discrete. Its transform \((X_p(e^{j\Omega}))\) is also periodic and discrete.

\[
x_p[n] = (x * p)[n]
\]

\[
X_p(e^{j\Omega})
\]
The weight of each impulse in the Fourier transform of a periodically extended function is $2\pi$ times the corresponding Fourier series coefficient.

\[ X_p(e^{j\Omega}) \]

\[ a_k \]

\[ -2\pi \quad -\frac{\pi}{4} \quad \frac{\pi}{4} \quad 2\pi \]

\[ -8 \quad -1 \quad 1 \quad 8 \quad \frac{1}{4} \]
Relation between Fourier Transforms and Series

The effect of periodic extension was to sample the frequency representation.

\[ X(e^{j\Omega}) \]

\[ \Omega \]

\[ a_k \]

\[ \frac{1}{4} \]
Four Types of “Time”

discrete vs. continuous (↑) and periodic vs aperiodic (↔)

**DT Fourier Series**

![DT Fourier Series Diagram]

**DT Fourier transform**

![DT Fourier transform Diagram]

**CT Fourier Series**

![CT Fourier Series Diagram]

**CT Fourier transform**

![CT Fourier transform Diagram]
Four Types of “Frequency”

discrete vs. continuous (↔) and periodic vs aperiodic (↕)

**DT Fourier Series**

![DT Fourier Series Diagram]

\[ \frac{2\pi k}{N} \]

**CT Fourier Series**

![CT Fourier Series Diagram]

\[ \frac{2\pi k}{T} \]

**DT Fourier transform**

![DT Fourier Transform Diagram]

\[ \Omega \]

**CT Fourier transform**

![CT Fourier Transform Diagram]

\[ \omega \]
Relation between Fourier Transforms and Series

Periodic extension of a DT signal produces a discrete function of frequency.

Periodic extension

= convolving with impulse train in time
= multiplying by impulse train in frequency
→ sampling in frequency

\[ N \to \infty \]

\[ T \to \infty \]
Relations among Fourier Representations

Different Fourier representations are related because they apply to signals that are related.

- **DTFS** (discrete-time Fourier series): periodic DT
- **DTFT** (discrete-time Fourier transform): aperiodic DT
- **CTFS** (continuous-time Fourier series): periodic CT
- **CTFT** (continuous-time Fourier transform): aperiodic CT

![Diagram showing the relations among Fourier representations]

- **DTFS** is related to **CTFS** through periodic extension and interpolation.
- **DTFT** is related to **CTFT** through periodic extension and interpolation.

The diagram illustrates the relationships and transformations connecting these representations.