Feedback and Control

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Feedback is pervasive in natural and artificial systems.

Turn steering wheel to stay centered in the lane.
Feedback and Control

Concentration of glucose in blood is highly regulated and remains nearly constant despite episodic ingestion and use.

Food enters the digestive system, which breaks down food into glucose. Glucose then enters the circulatory system. The pancreas (β cells) responds to glucose concentration, releasing insulin. Insulin acts on cells and tissues to lower glucose concentration in the blood. Glucose is also stored in cells and tissues for later use.
Today’s goal

Use systems theory to gain insight into how to control a system.
Example: wallFinder System

Approach a wall, stopping a desired distance $d_i$ in front of it.

$d_i = \text{desiredFront}$

$d_o = \text{distanceFront}$

What causes these different types of responses?
(Simple) Control systems have three parts.

The **plant** is the system to be controlled.
The **sensor** measures the output of the plant.
The **controller** specifies a command $C$ to the plant based on the *difference* between the input $X$ and sensor output $S$. 
Analysis of wallFinder System

Cast wallFinder problem into control structure.

proportional controller: \[ v[n] = K e[n] = K (d_i[n] - d_s[n]) \]

locomotion: \[ d_o[n] = d_o[n - 1] - TV[n - 1] \]

sensor with no delay: \[ d_s[n] = d_o[n] \]
Analysis of wallFinder System: Block Diagram

Visualize as block diagram.

\[ d_i = \text{desiredFront} \]

\[ d_o = \text{distanceFront} \]

proportional controller: \[ v[n] = Ke[n] = K(d_i[n] - d_s[n]) \]

locomotion: \[ d_o[n] = d_o[n - 1] - Tv[n - 1] \]

sensor with no delay: \[ d_s[n] = d_o[n] \]
Analysis of wallFinder System: System Function

Solve.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ D_i \quad + \quad K \quad V \quad -T \quad + \quad R \quad \rightarrow \quad D_o \]

\[
\frac{D_o}{D_i} = \frac{-KTR}{1-R} = \frac{-KTR}{1-R-KTR} = \frac{-KTR}{1-(1+KT)R}
\]
The system function contains a single pole at $z = 1 + KT$.

$$\frac{D_o}{D_i} = \frac{-KTR}{1 - (1 + KT)R}$$

Unit-sample response for $KT = -0.2$:

What determines the speed of the response? Could it be faster?
Check Yourself

Find $KT$ for fastest convergence of unit-sample response.

\[ \frac{D_o}{D_i} = \frac{-KTR}{1 - (1 + KT)\mathcal{R}} \]

1. $KT = -2$
2. $KT = -1$
3. $KT = 0$
4. $KT = 1$
5. $KT = 2$
0. none of the above
Check Yourself

Find $KT$ for fastest convergence of unit-sample response.

\[
\frac{D_o}{D_i} = \frac{-KTR}{1 - (1 + KT)R}
\]

If $KT = -1$ then the pole is at $z = 0$.

\[
\frac{D_o}{D_i} = \frac{-KTR}{1 - (1 + KT)R} = R
\]

Unit-sample response has a single non-zero output sample, at $n = 1$. 
Analysis of wallFinder System: Poles

The poles of the system function provide insight for choosing $K$.

\[
\frac{D_o}{D_i} = \frac{-KTR}{1-(1+KT)R} = \frac{(1-p_o)R}{1-p_oR} \quad ; \quad p_0 = 1 + KT
\]

- $0 < p_0 < 1$, monotonic converging
- $-1 < KT < 0$, alternating converging
- $-2 < KT < -1$, alternating converging
- $p_0 < -1$, alternating diverging
- $KT < -2$, alternating diverging
Find $KT$ for fastest convergence of unit-sample response.

\[
\frac{D_o}{D_i} = \frac{-KTR}{1 - (1 + KT)R}
\]

1. $KT = -2$
2. $KT = -1$
3. $KT = 0$
4. $KT = 1$
5. $KT = 2$
0. none of the above
Analysis of wallFinder System

The optimum gain $K$ moves robot to desired position in one step.

$d_i = \text{desiredFront} = 1 \text{ m}$

$d_o = \text{distanceFront} = 2 \text{ m}$

$KT = -1$

$K = -\frac{1}{T} = -\frac{1}{1/10} = -10$

$v[n] = K(d_i[n] - d_o[n]) = -10(1 - 2) = 10 \text{ m/s}$

exactly the right speed to get there in one step!
Analyzing wallFinder: Space-Time Diagram

The optimum gain $K$ moves robot to desired position in one step.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]
Analyzing wallFinder: Space-Time Diagram

The optimum gain $K$ moves robot to desired position in **one** step.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
The optimum gain $K$ moves robot to desired position in one step.

d_i = \text{desiredFront}
d_o = \text{distanceFront}

position

v = 10

time
The optimum gain $K$ moves robot to desired position in one step.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
\[ v = 0 \]
Analyzing wallFinder: Space-Time Diagram

The optimum gain $K$ moves robot to desired position in one step.

$d_i = \text{desiredFront}$

$d_o = \text{distanceFront}$

position

$v = 10$
$v = 0$
The optimum gain $K$ moves robot to desired position in one step.

$$d_i = \text{desiredFront}$$

$$d_o = \text{distanceFront}$$

$$v = 10$$
$$v = 0$$
$$v = 0$$
Analyzing wallFinder: Space-Time Diagram

The optimum gain $K$ moves robot to desired position in one step.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[
\begin{align*}
  & v = 10 \\
  & v = 0 \\
  & v = 0 \\
  & v = 0 \\
  & v = 0 \\
  & v = 0 \\
  & v = 0 \\
  & v = 0
\end{align*}
\]
Analysis of wallFinder System: Adding Sensor Delay

Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

proportional controller: \[ v[n] = Ke[n] = K(d_i[n] - d_s[n]) \]

locomotion: \[ d_o[n] = d_o[n - 1] - Tv[n - 1] \]

sensor with delay: \[ d_s[n] = d_o[n - 1] \]
Analysis of wallFinder System: Adding Sensor Delay

Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]

\[ d_o = \text{distanceFront} \]

\[ \text{position} \]

\[ \text{time} \]
Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
\[ v = 0 \]
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\[ d_i = \text{desiredFront} \]
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\[ v = -10 \]
Adding delay tends to destabilize control systems.

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\[ v = 10 \]
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Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
\[ v = 0 \]
\[ v = -10 \]
Analysis of wallFinder System: Adding Sensor Delay

Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]

\[ d_o = \text{distanceFront} \]

\[ v = 10 \]

\[ v = 0 \]

\[ v = -10 \]
Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

- \( v = 10 \)
- \( v = 0 \)
- \( v = -10 \)
- \( v = -10 \)
- \( v = 0 \)
Analysis of wallFinder System: Adding Sensor Delay

Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[
\begin{align*}
v &= 10 \\
v &= 0 \\
v &= -10 \\
v &= 0
\end{align*}
\]
Analysis of wallFinder System: Block Diagram

Incorporating sensor delay in block diagram.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

proportional controller: \[ v[n] = Ke[n] = K(d_i[n] - d_s[n]) \]

locomotion: \[ d_o[n] = d_o[n-1] - Tv[n-1] \]

sensor with no delay: \[ d_s[n] = d_o[n-1] \]

---

\[ D_i \quad + \quad K \quad V \quad -T \quad + \quad R \quad \]

\[ D_o \]
Find the system function \( H = \frac{D_o}{D_i} \).

1. \( \frac{KTR}{1 - R} \)
2. \( \frac{-KTR}{1 + R - KTR^2} \)
3. \( \frac{KTR}{1 - R} - KTR \)
4. \( \frac{-KTR}{1 - R - KTR^2} \)
5. none of the above
Check Yourself

Find the system function $H = \frac{D_o}{D_i}$.

Replace accumulator with equivalent block diagram.

\[
\frac{D_o}{D_i} = \frac{-KTR}{1 - \mathcal{R}} = \frac{-KTR}{1 - \mathcal{R} - KTR^2}
\]
Find the system function $H = \frac{D_o}{D_i}$.

1. $\frac{KTR}{1 - R}$
2. $\frac{-KTR}{1 + R - KTR^2}$
3. $\frac{KTR}{1 - R} - KTR$
4. $\frac{-KTR}{1 - R - KTR^2}$
5. none of the above
Substitute $\mathcal{R} \rightarrow \frac{1}{z}$ in the system functional to find the poles.

$$\frac{D_o}{D_i} = \frac{-KTR}{1 - \mathcal{R} - KTR^2} = \frac{-KT\frac{1}{z}}{1 - \frac{1}{z} - KT\frac{1}{z^2}} = \frac{-KTz}{z^2 - z - KT}$$

The poles are then the roots of the denominator.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$
If $KT$ is small, the poles are at $z \approx -KT$ and $z \approx 1 + KT$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT} \approx \frac{1}{2} \pm \sqrt{\left(\frac{1}{2} + KT\right)^2} = 1 + KT, -KT$$

Pole near 0 generates fast response.
Pole near 1 generates slow response.
Slow mode (pole near 1) dominates the response.
Feedback and Control: Poles

As $KT$ becomes more negative, the poles move toward each other and collide at $z = \frac{1}{2}$ when $KT = -\frac{1}{4}$.

$$z = \frac{1}{2} \pm \sqrt{(\frac{1}{2})^2 + KT} = \frac{1}{2} \pm \sqrt{(\frac{1}{2})^2 - \frac{1}{4}} = \frac{1}{2}, \frac{1}{2}$$

Persistent responses decay. The system is stable.
If $KT < -1/4$, the poles are complex.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT} = \frac{1}{2} \pm j \sqrt{-KT - \left(\frac{1}{2}\right)^2}$$

Complex poles $\rightarrow$ oscillations.
Same oscillation we saw earlier!

Adding delay tends to destabilize control systems.

\[ d_i = \text{desiredFront} \]
\[ d_o = \text{distanceFront} \]

\[ v = 10 \]
\[ v = 0 \]
\[ v = -10 \]
\[ v = -10 \]
\[ v = 0 \]
Check Yourself

What is the period of the oscillation?

1. 1  
2. 2  
3. 3  
4. 4  
5. 6  
0. none of above
Check Yourself

$KT = -1$

$\Im z \quad \text{z-plane} \quad \Re z$

$\frac{1}{2} \pm j \frac{\sqrt{3}}{2} = e^{\pm j\pi/3}$

$p_0^n = e^{\pm j\pi n/3}$

$\underbrace{e^{\pm j0\pi/3}, e^{\pm j\pi/3}, e^{\pm j2\pi/3}, e^{\pm j3\pi/3}, e^{\pm j4\pi/3}, e^{\pm j5\pi/3}}_{1}, \underbrace{e^{\pm j6\pi/3}}_{e^{\pm j2\pi} = 1}$
Check Yourself

What is the period of the oscillation?

1. 1  
2. 2  
3. 3  
4. 4  
5. 6  
0. none of above
The closed loop poles depend on the gain.

If $KT : 0 \rightarrow -\infty$: then $z_1, z_2 : 0, 1 \rightarrow \frac{1}{2}, \frac{1}{2} \rightarrow \frac{1}{2} \pm j\infty$
Find $KT$ for fastest response.

Closed-loop poles:

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$

1. 0  2. $-\frac{1}{4}$  3. $-\frac{1}{2}$
4. $-1$  5. $-\infty$  0. None of above
Check Yourself

\[ z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT} \]

The dominant pole always has a magnitude that is \( \geq \frac{1}{2} \).

It is smallest when there is a double pole at \( z = \frac{1}{2} \).

Therefore, \( KT = -\frac{1}{4} \).
Find $KT$ for fastest response.

The closed-loop poles are given by:

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$

Options for $KT$ are:

1. 0  
2. $-\frac{1}{4}$  
3. $-\frac{1}{2}$  
4. $-1$  
5. $-\infty$  
0. none of above
Destabilizing Effect of Delay

Adding delay in the feedback loop makes it more difficult to stabilize.

Ideal sensor: \(d_s[n] = d_o[n]\)

More realistic sensor (with delay): \(d_s[n] = d_o[n - 1]\)

Fastest response without delay: single pole at \(z = 0\).

Fastest response with delay: double pole at \(z = \frac{1}{2}\). much slower!
**Destabilizing Effect of Delay**

Adding more delay in the feedback loop is even worse.

More realistic sensor (with delay): \( d_s[n] = d_o[n - 1] \)

Even more delay: \( d_s[n] = d_o[n - 2] \)

Fastest response with delay: double pole at \( z = \frac{1}{2} \).

Fastest response with more delay: double pole at \( z = 0.682 \).

\( \rightarrow \) even slower
Feedback and Control: Summary

Feedback is an elegant way to design a control system.

Stability of a feedback system is determined by its dominant pole.

Delays tend to decrease the stability of a feedback system.