In-Class Problems Week 10, Wed.

Problem 1.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word \( BOOKKEEPER \).

(a) In how many ways can you arrange the letters in the word \( POKE \)?

(b) In how many ways can you arrange the letters in the word \( BO_1O_2K \)? Observe that we have subscripted the O’s to make them distinct symbols.

(c) Suppose we map arrangements of the letters in \( BO_1O_2K \) to arrangements of the letters in \( BOOK \) by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

\[
\begin{align*}
O_2BO_1K & \\
KO_2BO_1 & BOOK \\
O_1BO_2K & \quad OBOK \\
KO_1BO_2 & \quad KOBO \\
BO_1O_2K & \quad \quad \ldots \\
BO_2O_1K & \quad \quad \ldots
\end{align*}
\]

(d) What kind of mapping is this, young grasshopper?

(e) In light of the Division Rule, how many arrangements are there of \( BOOK \)?

(f) Very good, young master! How many arrangements are there of the letters in \( KE_1E_2PE_3R \)?

(g) Suppose we map each arrangement of \( KE_1E_2PE_3R \) to an arrangement of \( KEEPER \) by erasing subscripts. List all the different arrangements of \( KE_1E_2PE_3R \) that are mapped to \( REPEEK \) in this way.

(h) What kind of mapping is this?

(i) So how many arrangements are there of the letters in \( KEEPER \)?

(j) Now you are ready to face the BOOKKEEPER!

How many arrangements of \( BO_1O_2K_1K_2E_1E_2PE_3R \) are there?

(k) How many arrangements of \( BOOK_1K_2E_1E_2PE_3R \) are there?

(l) How many arrangements of \( BOOKKE_1E_2PE_3R \) are there?

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(m) How many arrangements of BOOKKEEPER are there?

Remember well what you have learned: subscripts on, subscripts off.
This is the Tao of Bookkeeper.

(n) How many arrangements of VOODOODOLL are there?

(o) How many length 52 sequences of digits contain exactly 17 two’s, 23 fives, and 12 nines?

Problem 2. (a) Show that the Magician could not pull off the trick with a deck larger than 124 cards.

Hint: Compare the number of 5-card hands in an $n$-card deck with the number of 4-card sequences.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of 124 cards.

Hint: Hall’s Theorem and degree-constrained (10.6.5) graphs.

Problem 3.
The Magician can determine the 5th card in a poker hand when his Assistant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assistant reveals the other 7 cards.

Problem 4.
Solve the following counting problems. Define an appropriate mapping (bijective or $k$-to-1) between a set whose size you know and the set in question.

(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

(b) Write a multinomial coefficient for the number of nonnegative integer solutions for the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8.$$  \hfill (1)

(c) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?