In-Class Problems Week 9, Wed.

Problem 1.
Recall that for functions \( f, g \) on \( \mathbb{N} \), \( f = O(g) \) iff
\[
\exists c \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \tag{1}
\]

For each pair of functions below, determine whether \( f = O(g) \) and whether \( g = O(f) \). In cases where one function is \( O() \) of the other, indicate the smallest nonnegative integer, \( c \), and for that smallest \( c \), the smallest corresponding nonnegative integer \( n_0 \) ensuring that condition (1) applies.

(a) \( f(n) = n^2, g(n) = 3n \).
\[
\begin{array}{lll}
  f = O(g) & YES & NO \\
  g = O(f) & YES & NO
\end{array}
\]
If YES, \( c = \ldots \), \( n_0 = \ldots \)

(b) \( f(n) = (3n - 7)/(n + 4), g(n) = 4 \).
\[
\begin{array}{lll}
  f = O(g) & YES & NO \\
  g = O(f) & YES & NO
\end{array}
\]
If YES, \( c = \ldots \), \( n_0 = \ldots \)

(c) \( f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n \).
\[
\begin{array}{lll}
  f = O(g) & YES & NO \\
  g = O(f) & YES & NO
\end{array}
\]
If yes, \( c = \ldots \), \( n_0 = \ldots \)

Problem 2.

(a) Define a function \( f(n) \) such that \( f = \Theta(n^2) \) and NOT \( f \sim n^2 \).

(b) Define a function \( g(n) \) such that \( g = O(n^2), g \neq \Theta(n^2) \) and \( g \neq o(n^2) \).

Problem 3.

False Claim.
\[
2^n = O(1). \tag{2}
\]

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.
Bogus proof. The proof by induction on $n$ where the induction hypothesis, $P(n)$, is the assertion (2).

**base case:** $P(0)$ holds trivially.

**inductive step:** We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all $n$. That is, the exponential function is bounded by a constant.

\[ \square \]

**Problem 4.**

Give an elementary proof (without appealing to Stirling’s formula) that $\log(n!) = \Theta(n \log n)$.

### Asymptotic Notations

Let $f, g$ be functions from $\mathbb{R}$ to $\mathbb{R}$.

- $f$ is **asymptotically equal to** $g$: $f(x) \sim g(x)$ iff $\lim_{x \to \infty} f(x)/g(x) = 1$.
- $f$ is **asymptotically smaller than** $g$: $f(x) = o(g(x))$ iff $\lim_{x \to \infty} f(x)/g(x) = 0$.
- for $f, g$ nonnegative, $f = O(g)$ iff $\limsup_{x \to \infty} f(x)/g(x) < \infty$, where
  \[ \limsup_{x \to \infty} h(x) := \lim_{x \to \infty} \lub_{y \geq x} h(y). \]
  An alternative, equivalent, definition is
  \[ f = O(g) \text{ iff } \exists c, x_0 \in \mathbb{R}^+ \forall x \geq x_0. f(x) \leq cg(x). \]
- Finally, $f = \Theta(g)$ iff $f = O(g)$ AND $g = O(f)$. 

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