Harmonic Sum Integral Method

Book Stacking

How far out?

One book

balances if center of mass over table
\[ \Delta - \text{overhang} \equiv \text{horizontal distance from \( n \)-book to \((n+1)\)-book} \]

centers of mass
\[ \Delta = \frac{1}{2} \frac{1}{n+1} = \frac{1}{2(n+1)} \]

**Book stacking summary**

\[ B_n \text{ ::= overhang of } n \text{ books} \]
\[ B_1 = 1/2 \]
\[ B_{n+1} = B_n + \frac{1}{2(n+1)} \]
\[ B_n = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) \]

**Harmonic Sums**

\[ H_n \text{ ::= } 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \]

\[ n^{th} \text{ Harmonic number} \]

\[ B_n = \frac{H_n}{2} \]

**Integral estimate for } H_n**

\[ H_n = \text{area of rectangles} \]
\[ > \text{area under } 1/(x+1) = \int_0^n \frac{1}{x+1} \, dx = \int_1^{n+1} \frac{1}{x} \, dx = \ln(n+1) \]
Book stacking

for overhang 3, need \( B_n \geq 3 \)
\[ H_n \geq 6 \]

integral bound: \( \ln(n+1) \geq 6 \)

so \( ok \) with \( n \geq \lceil e^6 - 1 \rceil = 403 \) books

actually calculate \( H_n \):

227 books are enough.

Book stacking

\( H_n \to \infty \) as \( n \to \infty \),
so overhang can be
as big as desired!

CD cases over the edge

43 cases high -- top 4 cases completely
off the table -- 1.8 or 1.9 case-lengths

Upper bound for \( H_n \)

\[ H_n < 1 + \int_{1}^{n} \frac{1}{x} \, dx \]
\[ = 1 + \ln(n) \]

Asymptotic bound for \( H_n \)

\[ \ln(n+1) < H_n < 1 + \ln(n) \]

\( H_n \sim \ln(n) \)
Asymptotic Equivalence

Def: \( f(n) \sim g(n) \)

\[ \lim_{n \to \infty} \left( \frac{f(n)}{g(n)} \right) = 1 \]

Example: \( (n^2 + n) \sim n^2 \)

pf:

\[ \lim_{n \to \infty} \frac{n^2 + n}{n^2} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right) = 1 \]

Team Problems

Problems 1–3