The Well Ordering Principle

Well Ordering principle
Every nonempty set of nonnegative integers has a least element.

Familiar? Now you mention it, Yes.
Obvious? Yes.
Trivial? Yes. But watch out:

Every nonempty set of nonnegative rationals has a least element.

NO!

Well Ordering Principle Proofs
To prove \( \forall n \in \mathbb{N}. P(n) \) using WOP:
• define set of counterexamples
  \[ C := \{ n \in \mathbb{N} \mid \text{NOT } P(n) \} \]
• assume \( C \) is not empty. By WOP, have minimum element \( m \in C \)
• Reach a contradiction somehow ...
  usually by finding \( c \in C \) with \( c < m \)

Well Ordered Postage
available stamps: 5¢ 3¢

Thm: Get any amount \( n \geq 8¢ \)
Prove by WOP. Suppose not.
Let \( m \) be least counterexample:
if \( m > n \geq 8 \), can get \( n¢ \).
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\[ m > 8: \]
\[ m > 9: \]
\[ m > 10: \]

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So \( m \geq 11 \). Now \( m > m-3 \geq 8 \) so can get \( m-3\$ \). But

\[ \text{contradiction!} \]

Geometric sums

\[ 1 + r + r^2 + r^3 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1} \]

Proof by WOP. Let \( m \) be smallest \( n \) with \( \neq \). But = for \( n = 0, \) so \( m > 0, \) and

\[ 1 + r + r^2 + r^3 + \ldots + r^{m-1} = \frac{r^m - 1}{r - 1} \]

Geometric sums

\[ 1 + r + r^2 + r^3 + \ldots + r^{m-1} = \frac{r^m - 1}{r - 1} \]

add \( r^m \) to both sides

\[ \text{LHS} = 1 + r + r^2 + r^3 + \ldots + r^{m-1} + r^m \]
\[ \text{RHS} = \frac{r^m - 1}{r - 1} + \frac{r^{m+1} - r^m}{r - 1} = \frac{r^{m+1} - 1}{r - 1} \]

so = at \( m \), contradicting \( \neq \): there is no counterexample.

Team Problems

Problems 1–3