Graph Connectivity

Trees

Every graph consists of separate connected pieces (subgraphs) called connected components.

The connected component of vertex \( v \) := \( \{ w | v \text{ and } w \text{ are connected} \} \)

So a graph is connected iff it has only 1 connected component.

Def: vertices \( v, w \) are \( k \)-edge connected if they remain connected whenever fewer than \( k \) edges are deleted.
**k-edge Connectedness**

1-edge connected

No path

1-edge connected

**Edge Connectedness**

2-edge connected

No path

2-edge connected

**k-edge Connectedness**

Def: A whole graph is *k*-edge connected iff every two vertices are *k*-edge connected.

3-edge connected

**Edge Connectedness**

Connectivity measures **fault tolerance** of a network: how many connections can fail without cutting off communication?

**k-edge Connectedness**

This whole graph is 1-edge connected
An edge is a **cut edge** if removing it from the graph disconnects two vertices.

*Cut Edges*

- **B** is a cut edge
  - Deleting B gives two components

- **A** is not a cut edge
  - Still connected with edge A deleted

So a connected graph is 2-edge connected iff it has **no cut edge**.
Cycles

A cycle is a path that begins and ends with the same vertex.

Path: \( v \rightarrow b \rightarrow w \rightarrow w \rightarrow a \rightarrow v \)

Also: \( a \rightarrow v \rightarrow b \rightarrow w \rightarrow w \rightarrow a \)

Simple Cycles

A simple cycle is a cycle of length \( >2 \) that doesn't cross itself.

Path: \( v \rightarrow a \rightarrow w \rightarrow v \)

Also: \( w \rightarrow a \rightarrow v \rightarrow w \)

Simple Cycles

Length \( >2 \) implies that going back and forth over an edge is not a simple cycle.

Cut Edges and Cycles

Lemma: An edge is a cut edge iff it is on a simple cycle.
A tree is a connected graph with no simple cycles.

equivalently:

A tree is a connected graph with every edge a cut edge.

More Trees

Other Tree Definitions

- graph with a unique simple path between any 2 vertices
- connected graph with n vertices and n-1 edges
- an edge-maximal acyclic graph

Team Problems

Problems 1 & 2