**Functional Approximation**

(Variables are scalar in this example)

\[ f(x) \approx \sum_{n=0}^{N} c_n \phi_n(x) + \Delta(x) \]

Figuring out \( \Delta(x) \) is similar to solving whole problem

Increase \( N \) until function converges

\( \{ \varphi_n(x) \} \) favorite set of functions

\( \{ \mathbf{v}_n \} \) favorite set of vectors

\[ \mathbf{w} \approx \sum_{n=0}^{N} c_n \mathbf{v}_n \quad N < M \]

\[ \mathbf{v}_n \in \{ \mathbb{R}^m \} \]

**Basis:** \( \mathbf{e}_i = \sum_{n=0}^{N} d_{i,n} \mathbf{v}_n \)

\[ \mathbf{w}_{\text{approx}} \approx \sum_{n=0}^{N} c_n \mathbf{v}_n = \sum_{i} a_i \mathbf{e}_i = \sum_{i,n} a_i d_{i,n} \mathbf{v}_n \]

\( \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \Rightarrow \text{orthonormal} \)

\( \mathbf{c} = \mathbf{a}^T \mathbf{D} \)

We want to do the same with functions. How do you take dot product?

Define \( \langle \varphi_m, \varphi_n \rangle = \int_\text{range of } x g(x) \varphi_n(x) \varphi_m(x) \text{dx} \)

“works”: \( \langle \varphi_m, \varphi_n \rangle = \delta_{mn} \)

1) We chose a basis \( \{ \varphi_n(x) \} \) and an inner product

orthonormal: \( \langle \varphi_m, \varphi_n \rangle = \delta_{mn} \)

2) We’re trying to solve \( \hat{f}(x) = q(x) \) \n
("In most problems, these are all vectors, unknown but that looks too scary to start with")
Look for solutions: \( f^{\text{unknown}}(x) \approx \sum c_n \phi_n(x) \)

\[
\int_a^b dx \ g(x) \phi_m^*(\lambda) [\hat{O} f(x)] = \int_a^b dx \ g(x) \phi_m^*(x) q(x)
\]

solution will depend on \( a, b, c_n, m \).

Range favorite

\[
F(a, b, c_n, m) = \psi(m, a, b)
\]

\[
F(c_n, m) = \psi(m)
\]

Now solve for \( c_n \).

If \( \hat{O} \) is a linear operator:

\[
\hat{O} f^{\text{approx}}(x) = \hat{O} \sum c_n \phi_n(x) = \sum c_n (\hat{O} \phi_n)
\]

and if \( \hat{O} \phi_n = \lambda_n \phi_n \) (i.e. \( \phi_n \) is an eigenfunction of \( \hat{O} \))

\[
\hat{O} f^{\text{approx}}(x) = \sum c_n \lambda_n \phi_n(x)
\]

\[
\int_a^b dx \ g(x) \phi_m^* \sum c_n \lambda_n \phi_n = \sum c_n \lambda_n \int_a^b dx \ g(x) \phi_m^* \phi_n
\]

\[
\int_a^b dx \ g(x) \phi_m^* \hat{O} f^{\text{approx}} = \sum c_n \lambda_n \delta_{mn} = c_m \lambda_m
\]

\[
c_m = \frac{1}{\lambda_m} \int_a^b dx \ g(x) \phi_m^* (x) q(x)
\]

\[
\hat{O} = \left[ k \frac{\partial^2}{\partial x^2} + h(x) \right] T(x)
\]

Often this is the operator

\[
\sin \quad \cos
\]

are eigenfunctions

Gives you a really messy equation:

Suppose \( \hat{O} = \hat{O}_1 + h(x) \) \{i.e. Schrodinger Equation\}

Suppose \( \hat{O}_1 \phi_n = \lambda_n \phi \)
\[ \int_a^b dx \ g(x) \phi_m^* (x) \hat{\phi}^{approx} = c_m \lambda_m + \int_a^b dx \ g(x) \phi_m^* (x) h(x) \sum c_n \phi_n (x) \]

\[ \sum c_n \int_a^b dx \ g(x) \phi_m^* (x) h(x) \phi_n (x) \]

\[ H_{mn} \]

\[ c_m \lambda_m + \sum c_n H_{mn} = b_m \]

\[ (H+\Lambda)c = b \quad m=1,...,N \]

\[ \Lambda_m = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \]

Must evaluate integrals \( H_{mn} \): difficult to evaluate, quantum mechanics requires 6-dimensional integrals. \( H \) becomes a large matrix when \( n \) gets large.

Also have Boundary Conditions: \( f(x = 0) = f_0 \)

adds another equation:

\[ \sum c_n \phi_n (x = 0) = f_0 \]

\[ \forall \cdot c = f_0 \]

How to solve? Can try to fit by least squares and just fit all the equations approximately. Can drop larger \( n \) terms to leave space for boundary conditions. Another way would be to not consider the boundary conditions and then craftily choose \( \Phi_n \) so that they solve the boundary conditions.

To check if answer makes sense: write out the series and see if \( c_n \) converges

**Evaluate Residuals**

\[ R = \hat{\phi} f - q \]

\[ \max(R) < \text{tol?} \]

\[ ||R(x_i)|| < \text{tol?} \]

we will evaluate this later